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ABSTRACT

The study was undertaken to investigate the errors made by high and low achievers in a mathematics test administered to students in their second year of high school. The focus of the study was to determine whether any common error types existed within or between the two groups or whether there were error strategies which differed between the two groups. At the same time, the results of both groups were analyzed in the light of findings which indicated that high achievers tended to make non-systematic errors while low achievers made systematic errors. The subjects were students from five secondary high schools which had been participating in the Mt. Druitt Longitudinal Study. They had participated in programs aimed at bridging the educational achievement gap between "disadvantaged" students and those who were not "disadvantaged." The intervention programs ceased in 1979 when the children had completed first grade. Since that time an aspect of the study has been the assessment of educational skills of these students as they progressed through schooling. Results suggested that a difference did exist in the error patterns of the high and low achiever groups. (PK)

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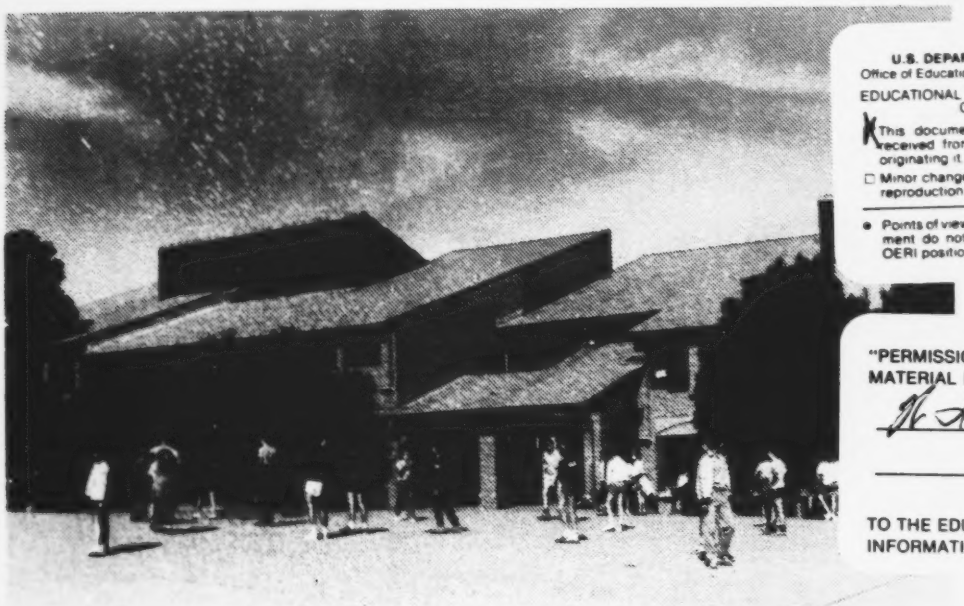
MT. DRUITT LONGITUDINAL STUDY

PATTERNS OF ERROR IN MATHEMATICS



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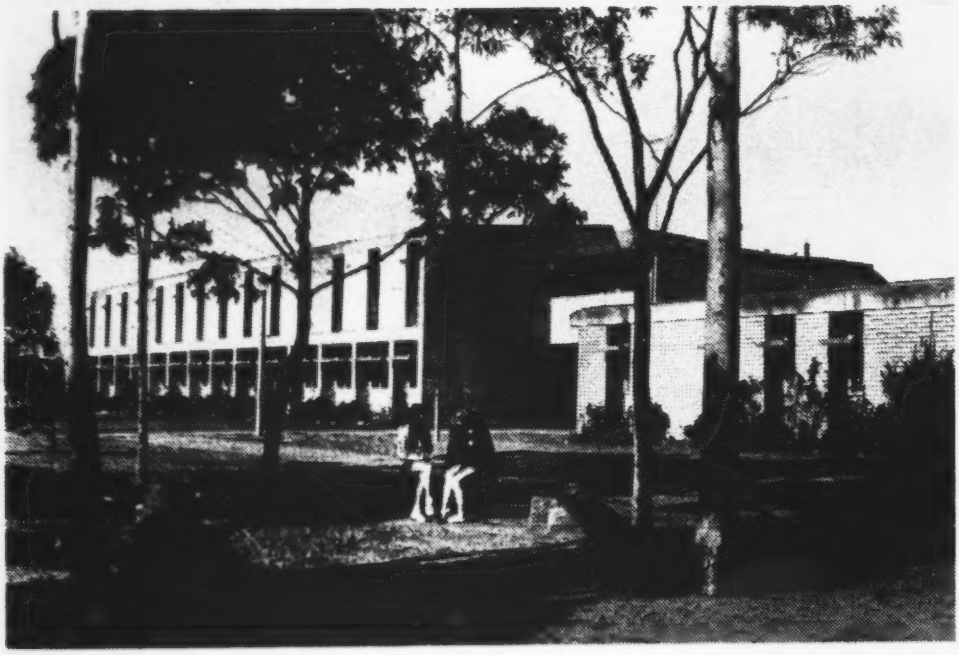

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Patterns of error in mathematics: A comparison of high achievers and low achievers

Abstract

Students errors in mathematics have been the focus of many recent research projects. This study of students in their second year of secondary schooling investigated the errors made by low and high achievers. The aim was to determine whether patterns of errors existed within and/or between the two groups. Analysis of responses to test items indicated that high achievers tended to make non-systematic errors whilst low achievers incorrectly adapted previously encountered strategies. For items unfamiliar to both groups, the same types of errors occurred.

INTRODUCTION

A great deal of recent research in mathematics teaching has focussed on the investigation of errors made by students. One of the findings from the concepts in Secondary Mathematics and Science project (C.S.M.S.) was that many secondary students were making the same error when solving certain test items (Hart, 1984). Other writers including Kieran (1980), Booth (1984), Burton (1980) and Bell, Costello and Kuchemann (1983) similarly have diagnosed common error types within various topics in secondary mathematics. All these writers suggest possible reasons for the error strategies ranging from lack of preparation (Hart, 1984) to the construction of incorrect algorithms used in solving particular problems (Burton, 1980).

The study reported here was undertaken to investigate the errors made by high and low achievers in a mathematics test administered to students in their second year of high school. The focus of the study was to determine whether any common error types existed within or between the two groups or whether there are error strategies which differ between the two groups. At the same time, the results of both groups were analysed in the light of the Australian findings by Clements (1982), that high achievers tend to make non-systematic errors whilst low achievers make systematic errors.

METHOD

Subjects

The subjects sitting for the test were Year 8 students from five secondary high schools, in Sydney, which had been participating in the Mt Druitt Longitudinal Study (O'Brien, 1981). The schools in this area, to the west of Sydney, had been identified, by welfare agencies, as being disadvantaged

and had participated in programs aimed at bridging the educational achievement gap between these "disadvantaged" students and those who were not "disadvantaged". The intervention programs ceased in 1979 when the children had completed first grade. Since that time an aspect of the Study has been the assessment of educational skills of these students as they progressed through schooling. To determine how, and to what extent, the students were disadvantaged, contrast schools were chosen to undergo the same testing as the Mt Druitt schools. The number of students in the two contrast high schools has reduced greatly, due largely to many of them going to private schools. In the three Mt Druitt schools 603 students were to be tested, whilst at the two contrast schools, a smaller sample of 90 students was to be tested, giving a total of 693 students. The High and Low Achievers were to be those who were, respectively, more than one standard deviation above or below the mean for the total sample.

Instrument

The instrument was compiled using fifty-four questions from the item bank of the ACER Maths Profile Series which is designed to measure students mathematical development throughout their schooling. This is achieved by a common scale of measurement whereby the items in the Series are grouped according to their difficulty and the students are grouped according to their ability. The age norms are based on experimental data collected by Collis (1975) & Wines (1975) and from ACER trial tests in a number of Victorian, New South Wales and Queensland schools. The norms are suggested as guidelines only because the trial schools, whilst from various backgrounds, are in no way a representative sample. The items chosen for inclusion were from four major content areas; Measurement, Operations, Number and Space. The questions were divided into two tests of 30 and 24

questions respectively which were to be administered on two separate occasions within the same school week. Each test was designed to be completed during a normal lesson of 35 to 40 minutes. Written instructions were given to the teachers for administering the tests. The number of students who completed the tests was 487 from the Mt Druitt schools and 79 from the contrast schools, giving a total sample of 566.

RESULTS & DISCUSSION OF RESULTS

The mean for the total sample was 27.00 with a standard deviation of 10.11. Using the selection criteria for high or low achievement produced the breakdown shown in Table 1.

TABLE 1. BREAKDOWN OF NUMBER OF HIGH AND LOW ACHIEVERS IN 1986 MATHS TESTING

	High achievers	Low achievers
Mt Druitt	56	96
Contrast	48	4
Total	104	100

The large number of high achievers, and the concomitant small number of low achievers, in the contrast sub-sample was to be expected given the nature of the composition of the schools in the Mt Druitt Longitudinal Study. The small number of low achievers in the contrast sub-sample renders comparison of error types between the sub-samples for this sub-group impossible. Hence, only a description of error types over this whole sub-group will be made. For the high achievers it is possible to compare with and across the two sub-samples.

With respect to differences in error patterns between the two high achieving groups analysis revealed no difference in the error pattern. There was the predictable higher indicence of errors in the Mt Druitt group, however, with the exception of one question, there was homogeneity in the error type in the results for the two groups. Thus analysis of error patterns, consequently, will focus here on the differences between the high and low achievers. Prior to examination, the types of errors and the spread of responses of the two groups may be illustrated. In Table 2 is shown a breakdown of the incorrect responses made by the students.

Apart from the obvious difference in the number of errors made by each group, two features emerge from Table 2. One is the larger proportion of students in the "No Response" category for the Low Achievers and the other is the greater spread of incorrect choices within each question by the Low Achievers. This latter feature contrasts with the errors of the High Achievers where, for the most part, a particular response was chosen by a large proportion of the students. In thiry-seven cases, at least 50% of the high achieving students making an error chose the same incorrect response compared to only five cases for the low achieving students. If the questions in which few errors were made were excluded from this analysis, the number would still be high.

TABLE 2. RESPONSES CHOSEN AS PERCENTAGE OF TOTAL ERRORS

	HIGH ACHIEVERS (n=104)							LOW ACHIEVERS (n=100)						
Question	no. of errors	A	B	C	D	E	N/R	no. of errors	A	B	C	D	E	N/R
1	2	50	50					49	35	37		24		4
2	5	20			80			73	45		32	21		2
3	4	75			25			62	35		19	40		6
4	2	50					50	51	24	37	31			8
5	14		7	43	50			75		18	29	45		8
6	19	17	78	5				71	46	24	23			7
7	24	4	9		87			64	25	30		34		11
8	19		33	10	38		19	79		29	35	26		10
9	51			12	45	37		93	11		10	44	25	10
10	21	43		33	19		5	76	21		38	29		12
11	23		9	18	64		9	76		17	8	61		14
12	32	32	13	55				90	28	31	30			11
13	17	29	35	29			17	72	22	26	32			20
14	27	12	27		61			67	12	33		31		24
15	34	3	82	15				83	30	35	22			13
16	6	33		17	50			80	34		39	14		13
17	15	33		53	14			60	14		40	33		13
18	14	64		36				84	5		40	40		15
19	19	16		68	11		5	72	31		29	21		19
20	4		25	75				74	19	14	50			17
21	11	9		82	9			64	19		28	31		22
22	23	9	91					68	13	37	26			24
23	11	100						73	47	12		18		23
24	4		75		25			77		26	32	16		26
25	15	67		7	26			81	25		30	26		19
26	12	17	25	58				88	31	40	7			21
27	32	48		29	19		4	91	9		41	31		19
28	31	70	10		10		10	84	23	23		33		21
29	26		80		8		12	69		27	33	14		26
30	31	19	29	39			13	85	31	32	15			22
31	9	33		22	45			73	27		33	36		4
32	17		47	41	12			61		64	18	18		
33	10	10		20	70			76	22		33	39		6
34	8		100					51	18	59		23		
35	32	74	6		20			76	41	37		21		1
36	15		27	73				26		31	46	23		
37	21		95		5			76	17	74		9		
38	11	25	50	8			17	73	29	45	25			1
39	40	34	11	55				74	26	18	56			
40	21	29	52		19			63	27	39		30		4
41	44	30	30		19	16	5	83	23	14		25	32	6
42	26	20	20	56			4	61	21	33	39			7
43	12	50	25		25			78	24	32		37		7
44	8	25	12	50			13	83	30	28	36			6
45	26	4	27	35		31	3	81	17	25	35		19	4

TABLE 2. (cont.)

	HIGH ACHIEVERS (n=104)							LOW ACHIEVERS (n=100)						
Question	no. of errors	A	B	C	D	E	N/R	no. of errors	A	B	C	D	E	N/R
46	15	7		33	47		13	76	30		28	34		8
47	32	3		39	13	45		76	18		25	16	30	11
48	35	34	26		31		9	78	41	32		19		8
49	40	90	5		3		2	87	45	28		21		6
50	35		37	34	20		9	84		42	20	29		8
51	27		35	50			15	81		49	41	1		9
52	21			33	48		19	78	29		32	31		8
53	55	40	26		25		9	79	28	43		20		9
54	29	11	46	25			18	82	28	43	17			12

Mathematical Operations

The questions examined here will be those for which the proportion of error response for one group is markedly different to the other, for example:

Question 18 $7 - 4 = \Delta - 7$.

In this question, 64% of the errors for the High Achievers were for the response $\Delta = 11$. This may be an example of what Clements (1982) would term a "non-systematic" (careless) error. Only 5% of the errors for the Low Achievers were due to this response. The Low Achievers major errors for this question were $\Delta = 4$ (40%) and $\Delta = 3$ (40%). These choices are consistent with that group's choices for Question 27, which is of the same structure as Question 18. In Question 27, in response to the problem:

$654 - 543 = \Delta - 654$;

41% and 31% of the errors made by the Low Achievers, were respectively, the choices $\Delta = 543$ and $\Delta = 111$. The choices of $= 4$ (Question 18) and $\Delta = 543$ (Question 27) would appear to be an over-generalisation of the Commutative Law of Addition to the area of subtraction. This mirrors the finding reported by Bell, O'Brien and Shiu (1980) that 14 year olds treated

subtraction as commutative especially if large numbers were involved. Alternatively, the strategy may have been to match the numbers on both sides as Collis (1975) had found earlier. This type of error was not prevalent amongst the High Achievers.

The choices of $\Delta = 3$ (Question 18) and $\Delta = 11$ (Question 27) suggest that the student has performed the calculation on the left hand side of the equation and ignored the extra term on the right hand side. Collis (1975) suggests that this strategy is related to the cognitive development of the students who, in these two cases, were unable to perform the two operations necessary for solving the problem. Consequently, these students performed only part of the required calculation. Another explanation of this type of error is that the students may perceive the equality sign as an "operator" as opposed to an "equivalence" (Kieran 1980). To differentiate between these two strategies or to, perhaps, determine another strategy would necessitate interviews. What is clear, however, is that an error of significance is being made.

Question 17

$8 \div 4 = \Delta \div 8;$

produced the same error for the Low Achievers, where 33% of the error was accounted for by the response $\Delta = 2$. Whilst the High Achievers did not generalise the Commutative Law of Addition to subtraction, the same was not the case for multiplication and division, where in Question 17, 53% of the errors were accounted for by the response $\Delta = 4$.

It is of interest to note that, for the High Achievers in particular, the number of errors in the Operations content area increased when the numbers

did not occur for the Low Achievers as can be seen in Table 3 although their error rate, even on items with small numbers far outstripped the rate of the High Achieving group.

TABLE 3. ANALYSIS OF QUESTIONS WITH SAME STRUCTURE INVOLVING SMALL AND LARGE NUMBERS

Question	Item	Number of Errors: High Achievers	Low Achievers
16	$15 \div 5 = 30 \div \Delta$	6	80
29	$240 \div 15 = 480 \div \Delta$	26	69
18	$7 - 4 = \Delta - 7$	14	84
27	$654 - 543 = \Delta - 654$	32	91
26	$4 + 5 = (4 + 6) + (5 + \Delta)$	12	88
30	$123 + 456 = (123 + 789) + (456 + \Delta)$	31	85

The error rate for the High Achievers group supports the findings of Collis (1972) and those of the Concepts in Secondary Maths and Science Project (CSMS) reported by Bell, Costello and Kuchemann (1983). Clearly, though, their findings are not supported in the case of the Low Achievers.

Collis (1972) found that the error rate increased further when pronumerals were introduced. There were two questions of the same structure in which pronumerals occurred (see Table4).

TABLE 4. ANALYSIS OF QUESTIONS WITH SAME STRUCTURE INVOLVING NUMBERS AND PRONUMERALS

Question	Item	Number of Errors: High Achievers	Number of Errors: Low Achievers
19	$(87 + 67) + 33 = \Delta + (67 + 33)$	19	72
24	$(h + j) + k = \Delta + (j + k)$	4	77
20	$769 \times 0 =$	4	74
23	$w \times 0 =$	11	73

Collis' results are not completely supported except in the second case by the High Achievers and, to a lesser extent, in the first case by the Low Achievers.

Questions 20 to 23 inclusive, illustrate a difference in the error patterns between the two groups. Whilst the occurrence of errors in these four questions indicates a misunderstanding of operations involving zero and unity, the High Achievers group was not as variable in its choice of the incorrect response as were the Low Achievers. In fact, for Question 23:

$w \times 0 = \Delta$;

the High Achievers were either correct or chose $\Delta = w$ as their response. The Low Achievers showed a preference for this response however, there remained a large proportion of students errors in the other two responses as well as in the "No Response" category.

Apart from questions on Mathematical Operations, the other content areas investigated in the testing were spatial relations, measurement and number. The responses in these three areas are not as readily isolated into error types as those in the Operations area. Without actually diagnosing each

error response, it is possible to generalise that, for both groups of students, the error types were very similar in questions involving the interpretation of a diagram (Questions 5, 11-13, 31, 32, 34, 37-42 and 50). In the majority of these cases the proportion of each incorrect response for a question was similar, or followed the same pattern, across the two groups. The exceptions were Questions 11, 34, 37 and 39 where one incorrect response was chosen in more than 50% of the cases, whilst in the other cases there was a spread of incorrect responses. It would be worthwhile to analyse these four questions. (Figure 1).

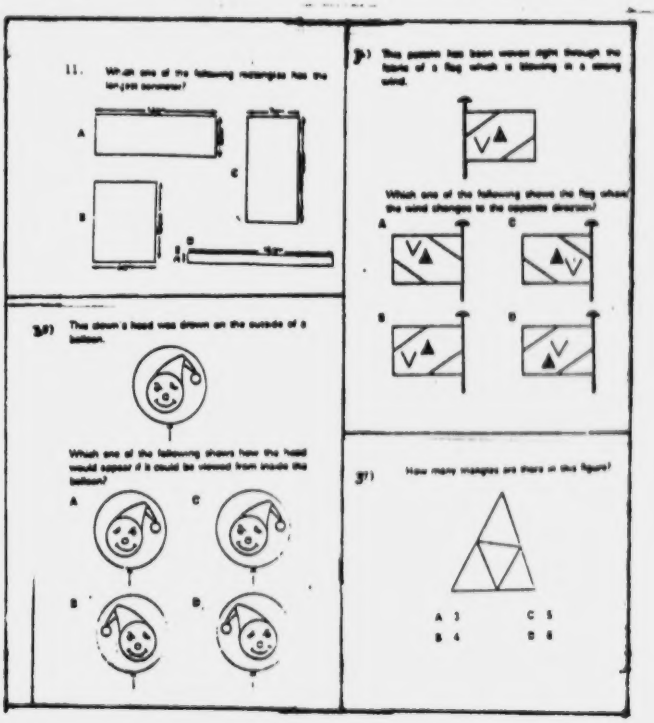

Figure 1.

Items 34 and 39 are very similar in that both test an understanding of spatial relationships, and it is of interest that the High Achievers tended to make the same error in both cases (Questions 34 - 100% had response C; Question 39 - 55% had response C).

For Question 37, the common error for both groups was that the large triangle was not counted. The dominating error for Question 11 (Response D) appears to have occurred, once again, through visual interpretation of the diagram where, in this case, the students may have regarded that the extra length of response D compensated for the reduced width. That the diagrams are not to scale may have been a contributing factor to the errors which did occur.

There were two Measurement questions (6 and 7) in which the errors of the two groups differed markedly. (see Figure 2.)

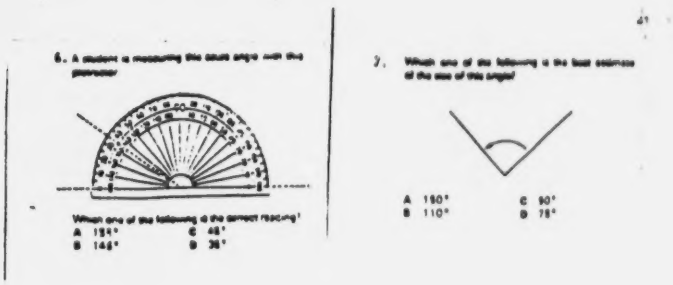

Figure 2.

For Question 6, the High Achievers error is due to either their not knowing the difference between "acute" and "obtuse" or in misreading the question. The Low Achievers, on the other hand, in choosing the responses 45° and 155° demonstrated their inability to correctly use the measuring device. Question 7, involving the estimation of an angle size, showed that the Low Achievers were unaware of the conventions for measuring angles when they gave A or B as responses. The majority of the errors in the High Achievers group (87%) demonstrated poor estimation ability.

With respect to items which could be described as "word" problems in which

there were large numbers of errors (Questions 8-10, 14, 15, 44, 49, 52-54), three of these (Questions 14, 15 and 49) revealed noticeably different types of errors. (see Figure 3.)

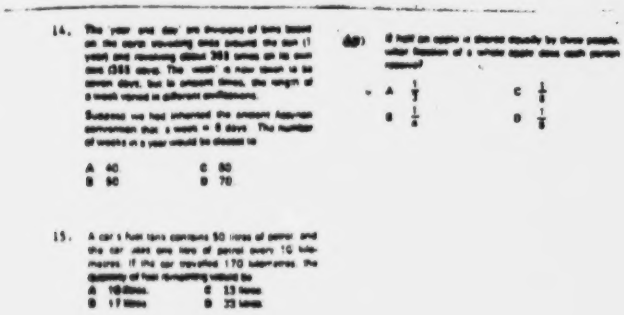

Figure 3.

In Question 14, 61% of the High Achievers errors came from choosing D, a choice which could be labelled a "careless" mistake. The Low Achievers showed a greater spread in their responses to Question 15, in which 82% of the High Achiever's errors came from choosing A, an error most likely due to incorrect reading of the question which led to calculating the quantity "used" rather than "remaining". Again the Low Achievers have a greater spread of errors. A similar finding occurred in Question 49 where 90% of the errors for the High Achievers was the response A, which could also be interpreted as a misreading of the question, whilst the Low Achievers had a spread of errors.

CONCLUSION

The purpose of this study was to determine if different error patterns occurred for High and Low Achievers. The fact that the question which High Achievers found the most and the least difficult were not the same as for the Low Achievers suggests that there does exist a difference in the error patterns of the two groups. Possible explanations for these differences may be given. The two groups encountered difficulty when interpreting

diagrams in both spatial and measurement problems. This study has found that, whilst the number of errors was not the same for these questions, the same types of errors were made by both groups. The writers suggest that these errors may be a function of the inexperience of the students with these types of questions and this subject matter. It is interesting to note that, in these cases, naive High Achievers do as poorly as naive Low Achievers.

In respect of the so-called word problems, interpretation led to errors for both groups. For these questions there appeared to be a difference in the error patterns in which a greater spread across the possible responses by the Low Achievers contrasted to the homogeneity of the High Achievers. It could be said that, for the latter group, incorrect reading of questions resulted in what have been described as non-systematic errors by Clements (1982). The Low Achievers, on the other hand, having demonstrated a larger spread in their responses, appeared to experience greater difficulty with comprehension. The large proportion of "No Response" for this group adds weight to this proposition. On this matter Bell et al (1983), on the basis of the CSMS Report, the APU Primary Survey and the National Assessment of Educational Progress, state that the difficulty involved in word problems is due to the great variety of contexts in which the problems are set. However, it is clear that the role of language in mathematics is, at the same time, widespread and important. It is essential that students can cope with questions involving language. The Cockcroft Report (1982 : 91) suggests that this could be best achieved through clear, unambiguous explanation, joint discussion and by encouraging students to talk and write about their work. Thus, far from minimising language in questions, greater

exposure to these "word" problems is recommended, especially for the Low Achievers who appear to have experienced greater difficulty with this type of question.

Responses to the items testing Mathematical Operations revealed a consistency in error patterns, especially for the Low Achievers who appeared to adopt and adapt strategies in responding. The errors of the High Achievers appeared to be of the non-systematic type referred to previously. The existence of what appears to be a strategy, especially for the Low Achievers, for solving these questions may be thought of as a blessing in disguise as these strategies can provide the necessary framework within which teachers can address the errors and adopt suitable strategies to overcome them. (The strategies which could be employed will not be discussed here as they lay beyond the scope of this paper.)

This study is unable to determine, without further investigation, whether any of the errors are due to the students' lack of pre-requisite knowledge. Nor has this study addressed what Carpenter (1984) suggests as an inherent problem in teaching (and hence testing) mathematics, namely, that the content order is premised on the logical order of topic presentation rather than on the developmental stages of the student. The writers are also cognisant of the Bell et al (1983) statement that it is difficult to determine the actual causes of errors and that it may be "nonsenical" to attempt to categorise, items because of the difficulties involved in classifying the responses. However, it is clear that certain types and patterns of error did occur for these two groups of students. Without attempting to generalise these results, it is clear that patterns of error existed and that they merit further investigation in an attempt to improve

our understanding of the processes students use in solving problems. As Booth (1984 : 158) succinctly states when discussing the errors made by the 14 and 15 year old students, a statement particularly appropriate to the errors of the Low Achievers group in this study: "(they are) not trivial mistakes based upon carelessness or idleness but rather (they) reflect definite points of view about the meaning of notation, the meaning of operations, and the nature of mathematical respresentations".

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